

A hybrid method for solving systems of operator inclusion problems

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Abstract

In this paper, we propose an algorithm combining the forward-backward splitting method and the alternative projection method for solving the system of splitting inclusion problem. We want to find a point in the interception of a finite number of sets that we don't know, the solution of each component of the system. The algorithm consists of approximate the sets involved in the problem by separates halfspaces which are a known strategy. By finding these halfspaces in each iteration we use only one inclusion problem of the system. The iterations consist of two parts, the first contains an explicit Armijo-type search in the spirit of the extragradient-like methods for variational inequalities. In the iterative process, the operator forward-backward is computed only one time for each inclusion problem, this represents a great computational saving because the computational cost of this operator is nothing cheap. The second part consists of special projection step, projecting in the separating halfspace. The convergence analysis of the proposed scheme is given assuming monotonicity all operators, without any Lipschitz continuity assumption.

Keywords: Armijo-type search, Maximal monotone operators, Forward-Backward, Alternative projection, Systems of inclusion problems, Armijo-type search

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1 Introduction

The goal of this paper is to present an algorithm for solving the system of inclusion problem, in which each component of the system is a sum of two operators, one point-to-set and the other point-to-point. Given a finite family of pair of operators $\{A_i, B_i\}_{i \in \mathbb{I}}$, with $\mathbb{I} =: (1, 2, \dots, m)$ and $m \in \mathbb{N}$. The system of inclusion problem consists in:

$$\text{find } x^* \in \mathbb{R}^n \text{ such that } 0 \in A_i(x^*) + B_i(x^*) \text{ for all } i \in \mathbb{I}, \quad (1)$$

where, for all $i \in \mathbb{I}$, the operators $A_i : \text{dom}(A_i) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ are point-to-point and maximal monotone and the operators $B_i : \text{dom}(B_i) \subset \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ are point-to-set maximal monotone operators. The solution of the problem, denoted by S_* , is given by the interception of the solution of each component of the system, i.e., $S_* = \cap_{i \in \mathbb{I}} S_*^i$, where S_*^i is defined as $S_*^i := \{x \in \mathbb{R}^n : 0 \in A_i(x) + B_i(x)\}$.

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Many problems in mathematics and science in general can be modeled as problem (1), for example, taking the operators $B_i = N_{C_i}$ with $C_i \subset \mathbb{R}^n$ convex sets for all $i \in \mathbb{I}$ we have the system of variational inequalities, introduced by I.V. Konnov in [16], which have been studied in [7–9, 14, 16, 17] and others. Some forward-backward algorithms for solving the inclusion problem, when the system contains just one equation, the hypothesis of Lipschitz continuity is very common see [12, 20]. In this paper, we improve this results assuming only maximal monotonicity for all operators A_i and B_i . Also, we improve the linesearch proposed by Tseng in [20], calculating only one time the forward-backward operator in each tentative to find the step size. Another advantage of the proposed algorithm is that in each iteration we not calculate the interception of any hyperplane like was do it in [10], and we use only one component of the system in each step of the algorithm, in the spirits of the alternative projection method. This improves the algorithm in the computational sense because any hard subproblem must be solved and because the forward-backward operator is very expensive to compute. The present work follows the ideas of the works [1, 4, 11].

Problem (1) have many applications in operations research, optimal control, mathematical physics, optimization and differential equations. This kind of problem has been deeply studied and has recently received a lot of attention, due to the fact that many nonlinear problems, arising within applied areas are mathematically modeled as nonlinear operator system of equations and/or inclusions, which each one is decomposed as a sum of two operators.

2 Preliminaries

In this section, we present some notation, definitions and results needed for the convergence analysis of the proposed algorithm. The inner product in \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$ and the norm induced by the inner product by $\|\cdot\|$. We denote by 2^C the power set of C . For X a nonempty, convex and closed subset of \mathbb{R}^n , we define the orthogonal projection of x onto X by $P_X(x)$, as the unique point in X , such that $\|P_X(x) - x\| \leq \|y - x\|$ for all $y \in X$. Let $N_X(x)$ be the normal cone to X at $x \in X$, i.e., $N_X(x) := \{d \in \mathbb{R}^n : \langle d, x - y \rangle \geq 0 \ \forall y \in X\}$. Recall that an operator $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is monotone if, for all $(x, u), (y, v) \in \text{Gr}(T) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in T(x)\}$, we have $\langle x - y, u - v \rangle \geq 0$, and it is maximal if T has no proper monotone extension in the graph inclusion sense. Now some known results.

Proposition 2.1 *Let X be any nonempty, closed and convex set in \mathbb{R}^n . For all $x, y \in \mathbb{R}^n$ and all $z \in X$ the following hold:*

- (i) $\|P_X(x) - P_X(y)\|^2 \leq \|x - y\|^2 - \|(P_X(x) - x) - (P_X(y) - y)\|^2$.
- (ii) $\langle x - P_X(x), z - P_X(x) \rangle \leq 0$.
- (iii) $P_X = (I + N_X)^{-1}$.

Proof. (i) and (ii) see Lemma 1.1 and 1.2 in [21]. (iii) See Proposition 2.3 in [3]. ■

In the following we state some useful results on maximal monotone operators.

Lemma 2.2 *Let $T : \text{dom}(T) \subseteq \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a maximal monotone operator. Then,*

- (i) $\text{Gr}(T)$ is closed.

(ii) T is bounded on bounded subsets of the interior of its domain.

Proof.

(i) See Proposition 4.2.1(ii) in [6].

(ii) Consequence of Theorem 4.6.1(ii) of in [6].

■

Proposition 2.3 *Let $T : \text{dom}(T) \subseteq \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a point-to-set and maximal monotone operator. Given $\beta > 0$ then the operator $(I + \beta T)^{-1} : \mathbb{R}^n \rightarrow \text{dom}(T)$ is single valued and maximal monotone.*

Proof. See Theorem 4 in [18].

■

Proposition 2.4 *Given $\beta > 0$ and $A : \text{dom}(A) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a monotone operator and $B : \text{dom}(B) \subseteq \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a maximal monotone operator, then*

$$x = (I + \beta B)^{-1}(I - \beta A)(x),$$

if and only if, $0 \in (A + B)(x)$.

Proof. See Proposition 3.13 in [13].

■

Now we define the so called Fejér convergence.

Definition 2.5 *Let S be a nonempty subset of \mathbb{R}^n . The sequence $(x^k)_{k \in \mathbb{N}} \subset \mathbb{R}^n$ is said to be Fejér convergent to S , if and only if, for all $x \in S$ there exists $k_0 \geq 0$, such that $\|x^{k+1} - x\| \leq \|x^k - x\|$ for all $k \geq k_0$.*

This definition was introduced in [5] and have been further elaborated in [15] and [1]. A useful result on Fejér sequences is the following.

Proposition 2.6 *If $(x^k)_{k \in \mathbb{N}}$ is Fejér convergent to S , then:*

- (i) *the sequence $(x^k)_{k \in \mathbb{N}}$ is bounded;*
- (ii) *the sequence $(\|x^k - x\|)_{k \in \mathbb{N}}$ is convergent for all $x \in S$;*
- (iii) *if a cluster point x^* belongs to S , then the sequence $(x^k)_{k \in \mathbb{N}}$ converges to x^* .*

Proof. (i) and (ii) See Proposition 5.4 in [2]. (iii) See Theorem 5.5 in [2].

■

3 The Algorithm

Let $A_i : \text{dom}(A_i) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B_i : \text{dom}(B_i) \subset \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be maximal monotone operators, with A_i point-to-point and B_i point-to-set, for all $i \in \mathbb{I}$. we assume that:

(A1) $\text{dom}(B_i) \subseteq \text{dom}(A_i)$, for all $i \in \mathbb{I} := \{1, 2, 3, \dots, m\}$ with $m \in \mathbb{N}$.

(A2) $S_* \neq \emptyset$.

(A3) For each bounded subset $V \subset \cap_{i=1}^m \text{dom}(B_i)$ there exists $R > 0$, such that $B_i(x) \cap B[0, R] \neq \emptyset$, for all $x \in V$ and $i \in \mathbb{I}$.

Where $B[0, R]$ is the closed ball centered in 0 and radius R . We emphasize that this assumption holds trivially if $\text{dom}(B_i) = \mathbb{R}^n$ or $V \subset \text{int}(\text{dom}(B_i))$ or B_i is the normal cone in any subset of $\text{dom}(B_i)$ for all $i \in \mathbb{I}$, i.e., in the application to system of variational inequality problem, this assumption is not necessary.

Choose any nonempty, closed and convex set, $X \subseteq \cap_{i \in \mathbb{I}} \text{dom}(B_i)$, satisfying $X \cap S_* \neq \emptyset$. The explanation for the chosen of X can be found in [4, 11, 20]. Let $(\beta_k)_{k=0}^\infty$ be a sequence such that $(\beta_k)_{k \in \mathbb{N}} \subseteq [\tilde{\beta}, \hat{\beta}]$ with $0 < \tilde{\beta} \leq \hat{\beta} < \infty$, and $\theta, \delta \in (0, 1)$, let $R > 0$ like Assumption (A3). The algorithm is defined as follows:

Algorithm A Let $(\beta_k)_{k \in \mathbb{N}}, \theta, \delta, R$ and \mathbb{I} like above.

Step 0 (Initialization): Take $x^0 \in X$.

Step 1 (Iterative Step 1): Given x^k , define $z_1^k := x^k$. Begin the process: for $i = 1$ to m do

$$J_i^k := (I + \beta_k B_i)^{-1}(I - \beta_k A_i)(z_i^k). \quad (2)$$

If $z_i^k = J_i^k$ put $i \in \mathbb{I}_k^*$ set $z_{i+1}^k = z_i^k$ and goto **Step 1**.

Stopping Criteria If $\mathbb{I}_k^* = \mathbb{I}$, then $x^k \in S_*$.

Step 1.1 (Inner Loop): Begin the inner loop over j . Put $j = 0$ and choose any $u_{(j,i)}^k \in B_i(\theta^j J_i^k + (1 - \theta^j) z_i^k) \cap B[0, R]$. If

$$\left\langle A_i(\theta^j J_i^k + (1 - \theta^j) z_i^k) + u_{(j,i)}^k, z_i^k - J_i^k \right\rangle \geq \frac{\delta}{\beta_k} \|z_i^k - J_i^k\|^2, \quad (3)$$

then $j_i(k) := j$ and stop. Else, $j = j + 1$. Define:

$$\alpha_{k,i} := \theta^{j_i(k)}, \quad (4)$$

$$\bar{u}_i^k := u_{j_i(k)}^k \quad (5)$$

$$\bar{x}_i^k := \alpha_{k,i} J_i^k + (1 - \alpha_{k,i}) x^k \quad (6)$$

$$z_{i+1}^k = P_X(P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k)). \quad (7)$$

Step 2 (Iterative Step 2): Define:

$$x^{k+1} := z_{m+1}^k, \quad (8)$$

set $k = k + 1$, empty \mathbb{I}_k^* and goto **Step 1**.

where

$$H_i(x, u) := \{y \in \mathbb{R}^n : \langle A_i(x) + u, y - x \rangle \leq 0\} \quad (9)$$

This method combine the Alternating Projection Method, the Forward-Backward Method and the ideas of separating hyperplane.

4 Convergence Analysis

In this section we analyze the convergence of the algorithms presented in the previous section. First, we present some general properties as well as prove the well-definition of the algorithm.

Lemma 4.1 *For all $(x, u) \in \text{Gr}(B_i)$, $S_*^i \subseteq H_i(x, u)$, for all $i \in \mathbb{I}$. Therefore $S_* \subset H_i(x, u)$ for all $i \in \mathbb{I}$.*

Proof. Take $x^* \in S_*^i$. Using the definition of the solution, there exists $v^* \in B_i(x^*)$, such that $0 = A_i(x^*) + v^*$. By the monotonicity of $A_i + B_i$, we have

$$\langle A_i(x) + u - (A_i(x^*) + v^*), x - x^* \rangle \geq 0,$$

for all $(x, u) \in \text{Gr}(B_i)$. Hence,

$$\langle A_i(x) + u, x^* - x \rangle \leq 0$$

and by (9), $x^* \in H_i(x, u)$. ■

From now on, $(x^k)_{k \in \mathbb{N}}$ is the sequence generated by the algorithm.

Proposition 4.2 *The algorithm is well-defined.*

Proof. The proof of the well-definition of $j_i(k)$ is by contradiction. Fix $i \in \mathbb{I} \setminus \mathbb{I}_k^*$ and assume that for all $j \geq 0$ having chosen $u_{(j,i)}^k \in B_i(\theta^j J_i^k + (1 - \theta^j) z_i^k) \cap B[0, R]$,

$$\left\langle A_i(\theta^j J_i^k + (1 - \theta^j) z_i^k) + u_j^k, z_i^k - J_i^k \right\rangle < \frac{\delta}{\beta_k} \|z_i^k - J_i^k\|^2.$$

Since the sequence $\{u_{(j,i)}^k\}_{j=0}^\infty$ is bounded, there exists a subsequence $\{u_{(\ell_j,i)}^k\}_{j=0}^\infty$ of $\{u_{(j,i)}^k\}_{j=0}^\infty$, which converges to an element u_i^k belonging to $B_i(z_i^k)$ by maximality. Taking the limit over the subsequence $\{\ell_j\}_{j \in \mathbb{N}}$, we get

$$\langle \beta_k A_i(z_i^k) + \beta_k u_i^k, z_i^k - J_i^k \rangle \leq \delta \|z_i^k - J_i^k\|^2. \quad (10)$$

It follows from (2) that

$$\beta_k A_i(z_i^k) = z_i^k - J_i^k - \beta_k v_i^k,$$

for some $v_i^k \in B_i(J_i^k)$.

Now, the above equality together with (10), lead to

$$\|z_i^k - J_i^k\|^2 \leq \left\langle z_i^k - J_i^k - \beta_k v_i^k + \beta_k u_i^k, z_i^k - J_i^k \right\rangle \leq \delta \|z_i^k - J_i^k\|^2,$$

using the monotonicity of B_i for the first inequality. So,

$$(1 - \delta) \|z_i^k - J_i^k\|^2 \leq 0,$$

which contradicts that $i \in \mathbb{I} \setminus \mathbb{I}_k^*$. Thus, the algorithm is well-defined. ■

Finally, a useful algebraic property on the sequence generated by the algorithm, which is a direct consequence of the inner loop and (6).

Corollary 4.3 Let $(x^k)_{k \in \mathbb{N}}$, $(\beta_k)_{k \in \mathbb{N}}$ and $(\alpha_{(k,i)})_{k \in \mathbb{N}}$ be sequences generated by the algorithm. With δ and $\hat{\beta}$ as in the algorithm. Then,

$$\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, z_i^k - \bar{x}_i^k \rangle \geq \frac{\alpha_{k,i}\delta}{\hat{\beta}} \|z_i^k - J_i^k\|^2 \geq 0, \quad (11)$$

for all k .

Proposition 4.4 If the algorithm stops, then $x^k \in S_*$.

Proof. If Stop Criteria is satisfied, then $\mathbb{I}_k^* = \mathbb{I}$ then, by Proposition 2.4 we have that $x^k \in S_*^i$ for all $i \in \mathbb{I}$ which imply that $x^k \in S_*$. ■

From now on assume that the algorithm generate an infinite sequence $(x^k)_{k \in \mathbb{N}}$.

Proposition 4.5 (i) The sequence $(x^k)_{k \in \mathbb{N}}$ is Fejér convergent to $S_* \cap X$.

(ii) The sequence $(x^k)_{k \in \mathbb{N}}$ is bounded.

(iii) For all $x^* \in S_* \cap X$ we have $\lim_{k \rightarrow \infty} \|z_j^k - x^*\|^2$ exist for all $j \in \mathbb{I}$ and satisfy that $\lim_{k \rightarrow \infty} \|z_j^k - x^*\|^2 = \lim_{k \rightarrow \infty} \|z_i^k - x^*\|^2$ for all $i, j \in \mathbb{I}$.

(iv) $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\|^2 = 0$.

Proof.

(i) Take $x^* \in S_* \cap X$. Using (7), (8), Proposition 2.1(i) and Lemma 4.1, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|z_{m+1}^k - x^*\|^2 = \|P_X(P_{H_m(\bar{x}_m^k, \bar{u}_m^k)}(z_m^k)) - P_X(P_{H_m(\bar{x}_m^k, \bar{u}_m^k)}(x^*))\|^2 \\ &\leq \|z_m^k - x^*\|^2 \leq \dots \leq \|z_1^k - x^*\|^2 = \|x^k - x^*\|^2. \end{aligned} \quad (12)$$

So, $\|x^{k+1} - x^*\| \leq \|x^k - x^*\|$.

(ii) Follows immediately from item (i).

(iii) Take $x^* \in S_* \cap X$. Using (12) yields for all $i \in \mathbb{I}$ that

$$\|x^{k+1} - x^*\|^2 \leq \|z_i^k - x^*\|^2 \leq \|x^k - x^*\|^2. \quad (13)$$

Now using Proposition 2.6 and item (ii) taking limits over k we have that $(\|z_i^k - x^*\|^2)_{k \in \mathbb{N}}$ is convergent for the same limits that $(\|x^k - x^*\|^2)_{k \in \mathbb{N}}$, independent of the $i \in \mathbb{I}$, obtaining the result.

(iv) Is a direct consequence of item (iii). ■

Proposition 4.6 For all $i \in \mathbb{I}$ we have,

$$\lim_{k \rightarrow \infty} \langle A_i(\bar{x}_i^k) + \bar{u}_i^k, z_i^k - \bar{x}_i^k \rangle = 0.$$

Proof. For all $i \in \mathbb{I}$. Using Proposition 2.1(i) and (7) for all $x^* \in S_* \cap X$ we have

$$\begin{aligned} \|z_{i+1}^k - x^*\|^2 &= \|P_X(P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k)) - P_X(P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(x^*))\|^2 \leq \|P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k) - P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(x^*)\|^2 \\ &\leq \|z_i^k - x^*\|^2 - \|P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k) - z_i^k\|^2. \end{aligned} \quad (14)$$

Now reordering (14), we get

$$\|P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k) - z_i^k\|^2 \leq \|z_i^k - x^*\|^2 - \|z_{i+1}^k - x^*\|^2.$$

Using the fact that,

$$P_{H(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k) = z_i^k - \frac{\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, z_i^k - \bar{x}_i^k \rangle}{\|A_i(\bar{x}_i^k) + \bar{u}_i^k\|^2} (A_i(\bar{x}_i^k) + \bar{u}_i^k),$$

and the previous equation, we have,

$$\frac{(\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, z_i^k - \bar{x}_i^k \rangle)^2}{\|A_i(\bar{x}_i^k) + \bar{u}_i^k\|^2} \leq \|z_i^k - x^*\|^2 - \|z_{i+1}^k - x^*\|^2. \quad (15)$$

By Proposition 2.3 and the continuity of A_i we have that J_i is continuo, since $(x^k)_{k \in \mathbb{N}}$ and $(\beta_k)_{k \in \mathbb{N}}$ are bounded then $\{J_i^k\}_{k \in \mathbb{N}}$, $(z_i^k)_{k \in \mathbb{N}}$ and $\{\bar{x}_i^k\}_{k \in \mathbb{N}}$ are bounded, implying the boundedness of $\{\|A_i(\bar{x}_i^k) + \bar{u}_i^k\|\}_{k \in \mathbb{N}}$ for all $i \in \mathbb{I}$.

Using Proposition 4.5(iii), the right side of (15) goes to 0, when k goes to ∞ , establishing the result. \blacksquare

Proposition 4.7 *For all $i \in \mathbb{I}$ we have $\lim_{k \rightarrow \infty} \|z_{i+1}^k - z_i^k\| = 0$.*

Proof. By definition of z_{i+1}^k and using that $z_i^k \in X$ for all $i \in \mathbb{I}$ and $k \in \mathbb{N}$ we have that

$$\|z_{i+1}^k - z_i^k\| = \|P_X(P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k)) - P_X(z_i^k)\| \leq \|P_{H_i(\bar{x}_i^k, \bar{u}_i^k)}(z_i^k) - z_i^k\|. \quad (16)$$

The right side of the equation (16) go to zero by Proposition 4.6, then the result follow. \blacksquare

A direct consequence of the previous proposition is that $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$, just summing and resting z_i^k for $i = 2, 3, \dots, m-1$ and using Cauchy-Swartz, we have the result. Other direct consequence of the Proposition 4.7 is that the sequences generated by the algorithm $(x^k)_{k \in \mathbb{N}}$, $(z_i^k)_{k \in \mathbb{N}}$ for each $i \in \mathbb{I}$ have the same clusters points.

Next we establish our main convergence result of the algorithm.

Theorem 4.8 *The sequence $(x^k)_{k \in \mathbb{N}}$ converges to some element belonging to $S_* \cap X$.*

Proof. Since $(x^k)_{k \in \mathbb{N}}$ is bounded then have cluster points, we claim that they belongs to $S_* \cap X$, as every x^k belong to X by definition and X is closed, then all clusters point of $(x^k)_{k \in \mathbb{N}}$ belong to X . The sequence $(x^k)_{k \in \mathbb{N}}$ is Fejér convergent to the set $S_* \cap X$, then by Proposition 2.6 (iii) the whole sequence will be convergent to this set. Let $(x^{j_k})_{k \in \mathbb{N}}$ be a convergent subsequence of $(x^k)_{k \in \mathbb{N}}$ such that, for all $i \in \mathbb{I}$ the sequences $(z_i^{j_k})_{k \in \mathbb{N}}$, $(\bar{x}_i^{j_k})_{k \in \mathbb{N}}$, $(\bar{u}_i^{j_k})_{k \in \mathbb{N}}$, $(\alpha_{j_k, i})_{k \in \mathbb{N}}$ and $(\beta_{j_k})_{k \in \mathbb{N}}$ are convergents, and as we see before as consequence of Proposition 4.7, calling the limits of $(x^{j_k})_{k \in \mathbb{N}}$ as \tilde{x} , we have

$\lim_{k \rightarrow \infty} x^{j_k} = \lim_{k \rightarrow \infty} z_i^{j_k} = \tilde{x}$, for all $i \in \mathbb{I}$.

Using Proposition 4.6 and taking limits in (11) over the subsequence $(j_k)_{k \in \mathbb{N}}$, we have for all $i \in \mathbb{I}$,

$$0 = \lim_{k \rightarrow \infty} \langle A_i(\bar{x}_i^{j_k}) + \bar{u}_i^{j_k}, z_i^{j_k} - \bar{x}_i^{j_k} \rangle \geq \lim_{k \rightarrow \infty} \frac{\alpha_{j_k, i} \delta}{\tilde{\beta}} \|x^{j_k} - J_i^{j_k}\|^2 \geq 0. \quad (17)$$

Therefore,

$$\lim_{k \rightarrow \infty} \alpha_{j_k, i} \|z_i^{j_k} - J_i^{j_k}\| = 0.$$

Now consider the two possible cases.

(a) First, assume that $\lim_{k \rightarrow \infty} \alpha_{j_k, i} \neq 0$, i.e., $\alpha_{j_k, i} \geq \bar{\alpha}$ for all k and some $\bar{\alpha} > 0$. In view of (17),

$$\lim_{k \rightarrow \infty} \|z_i^{j_k} - J_i^{j_k}\| = 0. \quad (18)$$

Since J_i is continuous, by the continuity of A_i and $(I + \beta_k B_i)^{-1}$ and by Proposition 2.3, (18) becomes

$$\tilde{x} = J_i(\tilde{x}, \tilde{\beta}) := (I + \tilde{\beta} B_i)^{-1} (I - \tilde{\beta} A_i)(\tilde{x}),$$

which implies that $\tilde{x} \in S_*^i$ for all $i \in \mathbb{I}$ using Proposition 2.4. Then $\tilde{x} \in S_*$ establishing the claim.

(b) On the other hand, if $\lim_{k \rightarrow \infty} \alpha_{j_k, i} = 0$ then for $\theta \in (0, 1)$ as in the algorithm, we have

$$\lim_{k \rightarrow \infty} \frac{\alpha_{j_k, i}}{\theta} = 0.$$

Define

$$y_i^{j_k} := \frac{\alpha_{j_k, i}}{\theta} J_i^{j_k} + \left(1 - \frac{\alpha_{j_k, i}}{\theta}\right) z_i^{j_k}.$$

Then,

$$\lim_{k \rightarrow \infty} y_i^{j_k} = \tilde{x}. \quad (19)$$

Using the definition of the $j_i(k)$ and (4), we have that $y_i^{j_k}$ does not satisfy (3) implying

$$\left\langle A_i(y_i^{j_k}) + u_{j_i(k)-1}^{j_k}, z_i^{j_k} - J_i^{j_k} \right\rangle < \frac{\delta}{\beta_{j_k}} \|z_i^{j_k} - J_i^{j_k}\|^2, \quad (20)$$

for $u_{j_i(k)-1}^{j_k} \in B_i(y_i^{j_k})$ and all $k \in \mathbb{N}$ and $i \in \mathbb{I}$.

Redefining the subsequence $\{j_k\}_{k \in \mathbb{N}}$, if necessary, we may assume that $\{u_{j_i(k)-1}^{j_k}\}_{k \in \mathbb{N}}$ converges to \tilde{u}_i . By the maximality of B_i , \tilde{u}_i belongs to $B_i(\tilde{x})$. Using the continuity of J_i , $(J_i^{j_k})_{k \in \mathbb{N}}$ converges to $J_i(\tilde{x}, \tilde{\beta})$ as defined in the first case. Using (19) and taking limit in (20) over the subsequence $(j_k)_{k \in \mathbb{N}}$, we have

$$\left\langle A_i(\tilde{x}) + \tilde{u}_i, \tilde{x} - J_i(\tilde{x}, \tilde{\beta}) \right\rangle \leq \frac{\delta}{\tilde{\beta}} \|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2. \quad (21)$$

Using the definition of $J_i(\tilde{x}, \tilde{\beta}) := (I + \tilde{\beta} B_i)^{-1} (I - \tilde{\beta} A_i)(\tilde{x})$ and multiplying by $\tilde{\beta}$ on both sides of (21), we get

$$\langle \tilde{x} - J_i(\tilde{x}, \tilde{\beta}) - \tilde{\beta} \tilde{u}_i + \tilde{\beta} \tilde{u}_i, \tilde{x} - J_i(\tilde{x}, \tilde{\beta}) \rangle \leq \delta \|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2,$$

where $\tilde{u}_i \in B_i(J_i(\tilde{x}, \tilde{\beta}))$. Applying the monotonicity of B_i , we obtain

$$\|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2 \leq \delta \|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2,$$

implying that $\|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\| \leq 0$. Thus, $\tilde{x} = J_i(\tilde{x}, \tilde{\beta})$ and hence, $\tilde{x} \in S_*^i$ for all $i \in \mathbb{I}$, thus $\tilde{x} \in S_*$. This prove the convergence of the whole sequence to the set $S_* \cap X$. \blacksquare

5 Conclusions

In this paper, we present an hybrid algorithm combining a variant of forward-backward splitting methods and the alternative projection method for solving a system of inclusion problems composed by the sum of two operators. A linesearch, for relax the hypothesis of Lipschitz continuity on forwards operators, have been proposed. The convergence analyze of the algorithm is proved. The results presented here, improve the previous in the literature by relaxing the hypothesis and the subproblems calculated here are computationally cheapest that knowing in the literature.

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